The Generalized Exponential Random Graph Model for Weighted Networks

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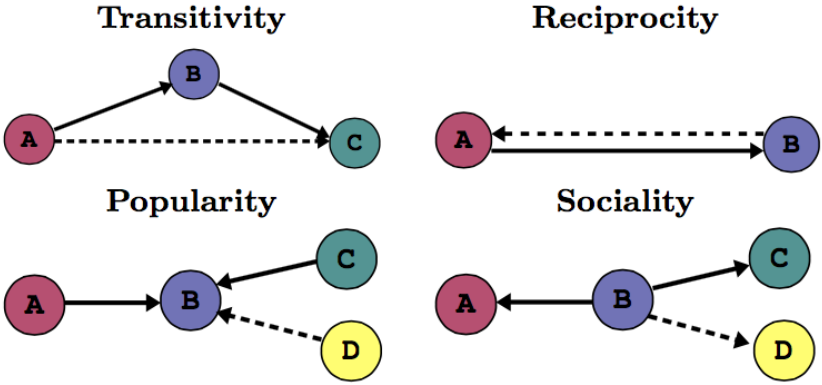
Outline

1. ERGMs for Unweighted Graphs
2. The Generalized Exponential Random Graph Model (GERGM)
3. Metropolis Hastings: Expanded Applicability
4. Likelihood Degeneracy
5. Software!

Assessing Network Structure



Exponential Random Graph Models (ERGMs)

Setting:

- Network x composed of n nodes and M edges
- Assume binary edges between nodes i, j: $x_{i,j} \in \{0, 1\}$
- X: family of *all* graphs with n nodes and binary edges
- $\mathbb{P}_X$: probability measure on X

Aim: Identify (and then estimate) the relational covariates that capture network structure through $\mathbb{P}_X$

Exponential Random Graph Models (ERGMs)

The probability (likelihood) of observing network x:

$$\mathbb{P}_X(x, \boldsymbol{\theta}) = \frac{\exp(\boldsymbol{\theta}^T \mathbf{h}(x))}{\sum_{z \in X} \exp(\boldsymbol{\theta}^T \mathbf{h}(z))}, \quad x \in \{0, 1\}^M$$

- $\mathbf{h} : \{0, 1\}^M \to \mathbb{R}^p$: Network covariates
- $\boldsymbol{\theta} \in \mathbb{R}^p$: unknown parameters (we have to estimate these!)

Weighted Graphs

Setting:

- Network y composed of n nodes and M edges
- Edges are continuous valued between nodes i, j: $y_{i,j} \in (-\infty, \infty)$
- Y: family of *all* graphs with n nodes
- $\mathbb{P}_Y$: probability measure on Y

Examples:

- Migration, Finance, Communication, Voting, Biological, Trade, etc.

Generalized Exponential Random Graph Model

Probability measure on family of weighted graphs w/ n nodes, M edges

Model: Two Steps

1) Model joint structure of Y on restricted network $X \in [0,1]^M$:

$$f_X(x, \boldsymbol{\theta}) = \frac{\exp\left(\boldsymbol{\theta}^T \mathbf{h}(x)\right)}{\int_{[0,1]^m} \exp\left(\boldsymbol{\theta}^T \mathbf{h}(z)\right) dz}, \quad x \in [0,1]^M$$

- $\mathbf{h} : [0,1]^M \to \mathbb{R}^p$: Network covariates (endogeneous or exogenous)
- $\boldsymbol{\theta} \in \mathbb{R}^p$: unknown structural parameters

Generalized Exponential Random Graph Model

2) Transform onto space of continuous weights:

$$f_Y(y,\theta,\Lambda) = \frac{\exp\left(\boldsymbol{\theta}^T \mathbf{h}(T(y,\boldsymbol{\beta}))\right)}{\int_{[0,1]^m} \exp\left(\boldsymbol{\beta}^T \mathbf{h}(z)\right) dz} \prod_{ij} t_{ij}(y,\boldsymbol{\beta}), \quad y \in \mathbb{R}^M$$

- $T : \mathbb{R}^M \to [0,1]^m$: parametric transformation function
  - Monotonically increasing
  - An appropriate choice: *cumulative distribution functions*
- $\boldsymbol{\beta} \in \mathbb{R}^q$: unknown transformation parameters

Features of the GERGM

- Flexible model for any type of weighted network
- Estimation via Markov Chain Monte-Carlo or Maximum pseudo-likelihood
- Simplifies to multivariate linear regression when $\mathbf{h}(x) = \mathbf{0}$

Model Specification: Weighted Network Statistics

Network Statistic	Parameter	Value
Reciprocity	θ_R	$\left(\sum_{i<j} x_{ij}x_{ji}\right)^{\alpha_R}$
Cyclic Triads	θ_{CT}	$\left(\sum_{i<j<k} (x_{ij}x_{jk}x_{ki} + x_{ik}x_{kj}x_{ji})\right)^{\alpha_{CT}}$
In-Two-Stars	θ_{ITS}	$\left(\sum_{i}\sum_{j<k\neq i} x_{ji}x_{ki}\right)^{\alpha_{ITS}}$
Out-Two-Stars	θ_{OTS}	$\left(\sum_{i}\sum_{j<k\neq i} x_{ij}x_{ik}\right)^{\alpha_{OTS}}$
Edge Density	θ_E	$\left(\sum_{i\neq j} x_{ij}\right)^{\alpha_E}$
Transitive Triads	θ_{TT}	$\left(\sum_{i<j<k} (x_{ij}x_{jk}x_{ik} + x_{ij}x_{kj}x_{ki} + x_{ij}x_{kj}x_{ik}) + \sum_{i<j<k} (x_{ji}x_{jk}x_{ki} + x_{ji}x_{jk}x_{ik} + x_{ji}x_{kj}x_{ki})\right)^{\alpha_{TT}}$

Inference on the GERGM

Likelihood of GERGM is intractable; relies on MCMC

Gibbs (Desmarais, et al., 2012)

Major Issue: Restricts model specification by requiring first order network statistics

$$\frac{\partial^2 \mathbf{h}(x)}{\partial x_{ij}^2} = \mathbf{0}, \quad i, j \in [n]$$

Metropolis-Hastings (Wilson, et al. 2015): Removes above restriction on GERGM

Metropolis-Hastings Sampling

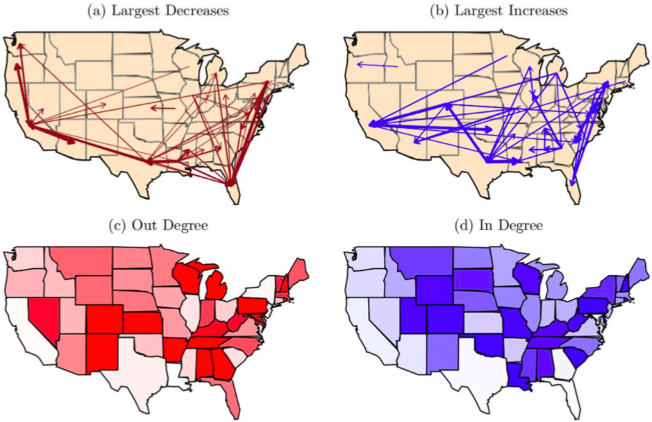
Framework: Acceptance/Rejection algorithm for weighted edges w/ multivariate truncated normal proposal distribution.

Proposal: $$q_\sigma(w|x) = \frac{\sigma^{-1}\phi(\frac{w-x}{\sigma})}{\Phi(\frac{1-x}{\sigma}) - \Phi(\frac{-x}{\sigma})}, \quad 0 \le w \le 1$$

Advantages:

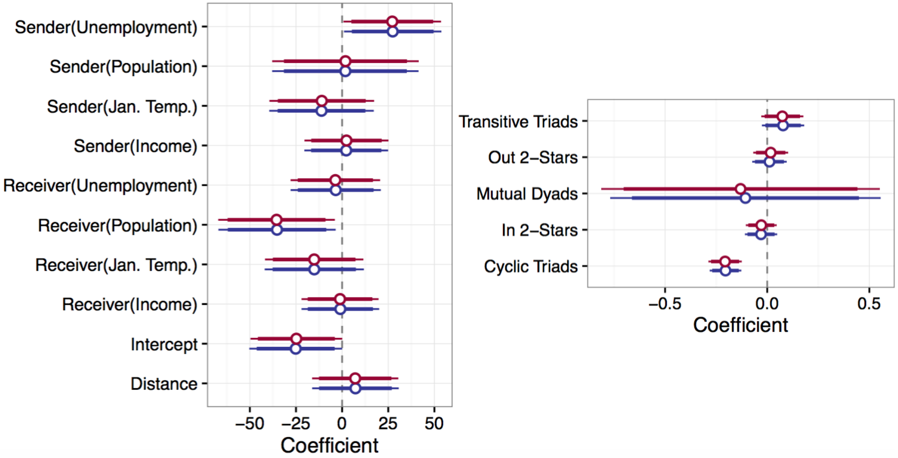
- Flexible model specification
  - Interaction effects
  - Exponential weighting of covariates
- New available models can avoid likelihood degeneracy

Comparison to Gibbs Sampling: U.S. Migration Data

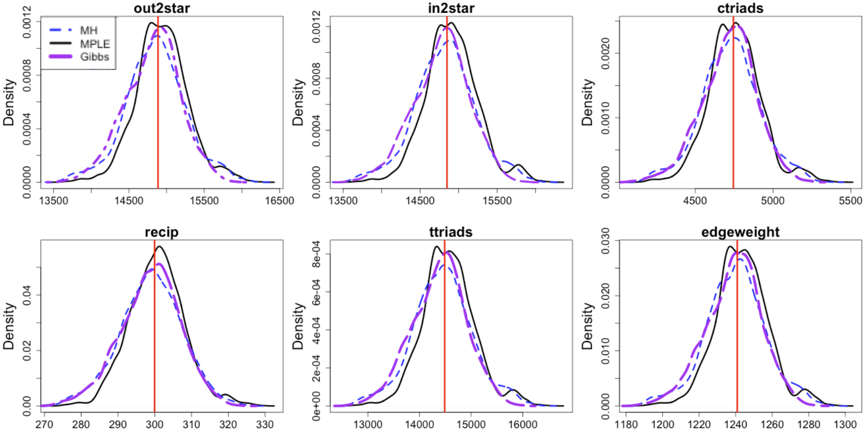


- Describes the inter-state migration in U.S. from 2006 to 2007.
- Fit a GERGM with 5 network statistics, 11 demographic covariates

Equivalent Covariate Estimates

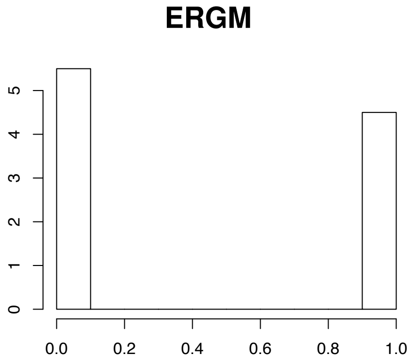


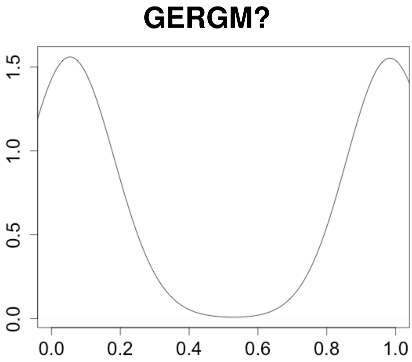
Goodness of Fit



- M-H and Gibbs have comparable (and good) performance

Likelihood Degeneracy





Case Study: In-Two-Stars Model

Model:

$$f_X(x, \theta, \alpha) = \frac{\exp\left(\theta_E h_E(x) + \theta_{ITS} h_{ITS}(x)\right)}{C(\theta_E, \theta_{ITS})}, \qquad x \in [0,1]^m$$

- Edge density: $h_E(x) = \sum_{i \neq j} x_{ij}/m$
- In-Two-Stars: $h_{ITS}(x, \alpha) = \left(\sum_i \sum_{j<k\neq i} x_{ji} x_{ki}\right)^{\alpha}$

Known to suffer from degeneracy issues in the binary case.

Exponential Down-Weighting

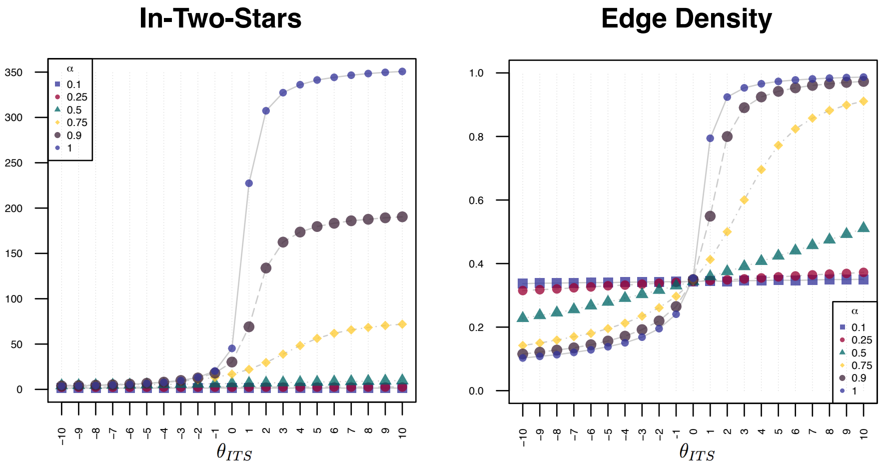
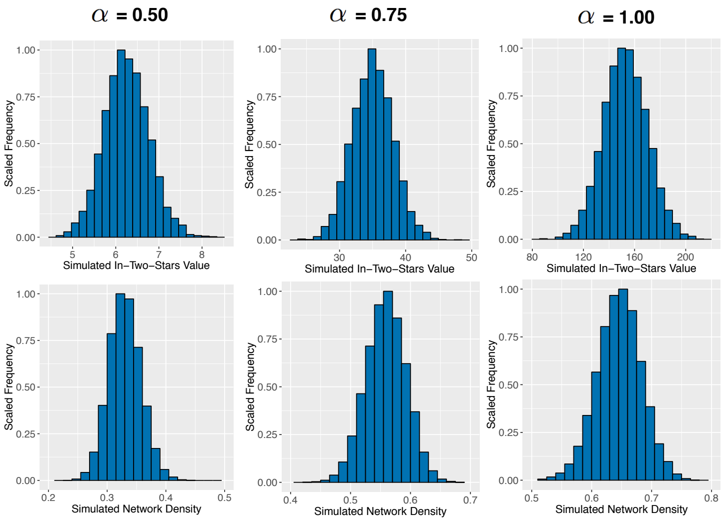


Figure: Statistics from 1M simulated In-Two-Stars networks with various values of α weighting.

Non-Degeneracy



Defeating Degeneracy

Causes:

- Bimodal density distribution vs. poor initialization.

Solutions:

- Estimate intercept via transformation.
- Exponential down-weighting.
- Weighted MPLE.
- Adaptive grid search.

GERGM Software in R

GERGM -- Master: build passing CRAN 0.7.4 Development: build passing

An R package to estimate Generalized Exponential Random Graph Models

PLEASE REPORT ANY BUGS OR ERRORS TO mdenny@psu.edu.

Model Overview

An R package which implements the Generalized Exponential Random Graph Model (GERGM) with an extension to estimation via Metropolis Hastings. The relevant papers detailing the model can be found at the links below:

- Bruce A. Desmarais, and Skyler J. Cranmer, (2012). "Statistical inference for valued-edge networks: the generalized exponential random graph model". PloS One. [Available Here]
- James D. Wilson, Matthew J. Denny, Shankar Bhamidi, Skyler Cranmer, and Bruce Desmarais (2015). "Stochastic Weighted Graphs: Flexible Model Specification and Simulation". [Available Here]

Installation

Requirements for using C++ code with R

See the **Requirements for using C++ code with R** section in the following tutorial: Using C++ and R code Together with Rcpp. You will likely need to install either `Xcode` or `Rtools` depending on whether you are using a Mac or Windows machine before you can use the package.

github.com/matthewjdenny/GERGM

Thank you!

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Materials:

- *Paper*: http://ssrn.com/abstract=2795219
- *Github*: github.com/matthewjdenny/GERGM
- *CRAN*: GERGM

GERGM - Metropolis Hastings Procedure

1. For $i, j \in [n]$, generate proposal edge $\tilde{x}_{i,j}^{(t)} \sim q_\sigma(\cdot|x_{i,j}^{(t)})$ independently across edges.
2. Set

$$x^{(t+1)} = \begin{cases} \tilde{x}^{(t)} = (\tilde{x}_{i,j}^{(t)})_{i,j\in[n]} & \text{w.p.} \quad \rho(x^{(t)}, \tilde{x}^{(t)}) \\ x^{(t)} & \text{w.p.} \quad 1 - \rho(x^{(t)}, \tilde{x}^{(t)}) \end{cases}$$

where

$$\rho(x, y) = \min\left(\frac{f_X(y|\theta)}{f_X(x|\theta)} \prod_{i=1}^{m} \frac{q_\sigma(x_i|y_i)}{q_\sigma(y_i|x_i)}, 1\right)$$

$$= \min\left(\exp\left(\theta'(\mathbf{h}(y) - \mathbf{h}(x))\right) \prod_{i=1}^{m} \frac{q_\sigma(x_i|y_i)}{q_\sigma(y_i|x_i)}, 1\right)$$