Advanced Network Analysis: Homework 1

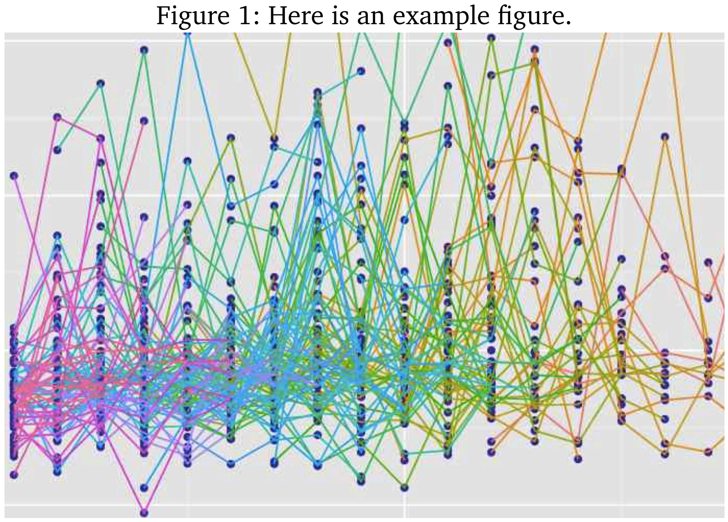
MATTHEW DENNY THURSDAY 23RD JULY, 2015

Question 1 : How do I add R code to my homework?

```
> p <- mean(rgamma(n=1000000,shape=20,scale=25) >= 150)
> print(p)
[1] 0.999992
> mean(rbinom(1000000, 100, p)>=75)
[1] 1
```

Question 2 : How do I add figures to my homework?

Figure 1: Here is an example figure.



Problem 1 : Derive Stirling's Formula

Stirling's formula is:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1}$$

(a) Argue that if $X_i \sim exponential(1), i = 1, 2, ...$, all independent, then for every x:

$$P\left(\frac{\overline{X}_n - 1}{1/\sqrt{n}} \leqslant x\right) \to P(Z \leqslant x) \tag{2}$$

where Z is a standard random normal variable.

We can make this argument on the basis of the central limit theorem. We recall the for of the exponential distribution:

$$f_x(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}} \tag{3}$$
$$E(X) = \beta \tag{4}$$
$$var(X) = \beta^2 \tag{5}$$

So for $\beta = 1$ we have $E[X] = 1$ and $Var(X) = 1$ thus we see that

$$P\left(\frac{\sqrt{n}(\overline{X}_n - 1)}{1} \leqslant x\right) \to P(Z \leqslant x) \tag{6}$$

which is true by the central limit theorem.

(b) Show that differentiating both sides in part (a) suggests:

$$\frac{\sqrt{n}}{\Gamma(n)}(x\sqrt{n}+n)^{n-1}e^{-(x\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{7}$$

and that $x = 0$ gives Stirling's Formula.

For the right hand side, we have:

$$\frac{d}{dx}P(Z \leqslant x) = \frac{d}{dx}F_Z(x) = f_Z(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{8}$$

which is the pdf of a standard normal distribution. Now for the left hand side we have:

$$P\left(\frac{\sqrt{n}(\overline{X}_n - 1)}{1} \leqslant x\right) = P\left(\sum_{i=1}^{n} X_i \leqslant x\sqrt{n}+n\right) \tag{9}$$

We now note that the exponential is a special case of the gamma distribution with $\alpha = 1$, and furthermore we note that the sum of gammas is also gamma so then we can define a new variable:

$$Y = \sum_{i=1}^{n} X_i \sim \text{gamma}(n, 1) \tag{10}$$

So then we have:

$$\frac{d}{dx}F_Y(x\sqrt{n}+n) = f_Y(x\sqrt{n}+n) \times \sqrt{n} = \sqrt{n}\frac{1}{\Gamma(n)}(x\sqrt{n}+n)^{n-1}e^{-(x\sqrt{n}+n)} \tag{11}$$

Therefore as $n \to \infty$ we have:

$$\frac{\sqrt{n}}{\Gamma(n)}(x\sqrt{n}+n)^{n-1}e^{-(x\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{12}$$

Now if we set $x = 0$

$$\frac{\sqrt{n}}{\Gamma(n)}(0\sqrt{n}+n)^{n-1}e^{-(0\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}}e^{-\frac{0}{2}} \tag{13}$$
$$\frac{\sqrt{n}}{\Gamma(n)}n^{n-1}e^{-n} \approx \frac{1}{\sqrt{2\pi}} \tag{14}$$
$$\sqrt{2\pi}\sqrt{n}n^{n-1}e^{-n} \approx \Gamma(n) \tag{15}$$
$$n \times \sqrt{2\pi}n^{n-\frac{1}{2}}e^{-n} \approx n \times (n-1)! \tag{16}$$
$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \approx n! \tag{17}$$

Table 1:

v	Exact	Normal Approx	Continuity Correction
0	0.0008	0.0071	0.0056
1	0.0048	0.0083	0.0113
2	0.0151	0.0147	0.0201
3	0.0332	0.0258	0.0263
4	0.0572	0.0392	0.0549
5	0.0824	0.0588	0.0664
6	0.1030	0.0788	0.0882
7	0.1148	0.0937	0.1007
8	0.1162	0.1100	0.1137
9	0.1085	0.1114	0.1144
10	0.0944	0.1113	0.1024

Which completes the derivation of Stirling's formula as desired.